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**The rate of metastable vacuum decay in (2+1)
dimensions**

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Abstract

The pre-exponential factor in the probability of decay of a metastable vacuum is calculated for a generic (2+1) dimensional model in the limit of small difference ϵ of the energy density between the metastable and the stable vacua. It is shown that this factor is proportional to $\epsilon^{-7/3}$ and that the power does not depend on details of the underlying field theory. The calculation is done by using the effective Lagrangian method for the relevant soft (Goldstone) degrees of freedom in the problem. Unlike in the (1+1) dimensional case, where the decay rate is completely determined by the parameters of the effective Lagrangian and is thus insensitive to the specific details of the underlying (microscopic) theory, in the considered here (2+1) dimensional case the pre-exponential factor is found up to a constant, which does depend on specifics of the underlying short-distance dynamics, but does not depend on the energy asymmetry parameter ϵ . Thus the functional dependence of the decay rate on ϵ is universally determined in the considered limit of small ϵ .

The problem of decay of a metastable vacuum state has attracted interest since long ago both in statistical physics[1, 2, 3, 4] and in the relativistic field theory[5, 6, 7, 8]. In the latter setting the metastable (false) vacuum is a state of one or more fields corresponding to a local rather than the global minimum of the energy density. Such state is stable under small quantum fluctuations of the field(s), however it can decay into a lower-energy vacuum state through large fluctuations described by the quantum tunneling. The mechanism of such transition[6, 7] is quite similar to that in the first order phase transitions and is described by nucleation and subsequent expansion of bubbles of the lower (true) vacuum amidst the bulk of the metastable phase. In the process of the expansion the excess ϵ of the energy density in the metastable vacuum (latent heat) is being converted into the energy of the (expanding) surface of the bubble. Clearly, the classical expansion of the bubble is possible only if the gain in the volume energy $\epsilon \cdot (Volume)$ compensates for the positive energy of the surface of the bubble $\mu \cdot (Surface\ Area)$ with μ standing for the surface tension. The minimal configuration where these are exactly equal is the critical spherical bubble with the radius $R_c = (d-1)\mu/\epsilon$, where d is the total number of the space and time dimensions. Starting from R_c the bubbles expand classically. The evolution in the classically forbidden domain at $R < R_c$ is described by the quantum tunneling.

It should be noted that in the above description the thickness of the surface is totally ignored in comparison with the radius of the bubble. The scale for the thickness of the transition region between the phases is set by the Compton wavelengths of the relevant particles in either of the vacua. Assuming that no relevant massless particles in either of the vacua are present in this problem, one readily concludes that the thin wall approximation is always valid in the limit of small ϵ which is assumed throughout the present paper.

The rate Γ of nucleation of a critical bubble per unit time and per unit volume is determined by the tunneling part of its trajectory in the classically forbidden domain, and the exponential factor, found from the action on this trajectory, is well known[6, 7] in terms of μ and ϵ :

$$\Gamma = F e^{-B} , \tag{1}$$

where $B = V_d (d-1)^{d-1} \mu^d / \epsilon^{d-1}$ with $V_d = \pi^{d/2} / \Gamma(1+d/2)$ being the volume of a unit ball in d dimensions. A calculation of the pre-exponential factor F requires a summation over the fluctuations of the field(s) near the tunneling trajectory[8]. In a (1+1) dimensional model, i.e. for $d = 2$, such summation can be done entirely within the thin wall approximation, and the factor F in eq.(1) is given[9] by $F = \epsilon/(2\pi)$. The latter expression is universal in the

sense that the only quantities that determine the rate Γ are μ and ϵ . In other words, any possible complexity of the underlying field dynamics reduces for the purpose of calculation of the rate Γ to two macroscopic parameters¹ μ and ϵ . The (1+1) dimensional expression for F fully agrees with calculations in specific field models[10, 11, 12].

The purpose of the present paper is to consider the problem of calculating the pre-exponential factor F in a (2+1) dimensional theory, i.e. for $d = 3$. Unlike in the (1+1) dimensional case, the knowledge of only the macroscopic parameters μ and ϵ is insufficient for a complete calculation of F for $d = 3$. However it will be shown that the dependence of the factor F on the parameter ϵ in the limit $\epsilon \rightarrow 0$ is still universal, while the dependence on (a combination of) the parameter μ and any mass scales in the underlying field theory, does depend on the details of the model (i.e. on the ‘microscopic’ dynamics). Thus some universality still remains in the (2+1) dimensional case, although in a substantially reduced form. Namely, it will be shown here that in (2+1) dimensions the rate of the false vacuum decay behaves at $\epsilon \rightarrow 0$ as

$$\Gamma = \frac{\mathcal{A}}{\epsilon^{7/3}} \exp\left(-\frac{16\pi}{3} \frac{\mu^3}{\epsilon^2}\right), \quad (2)$$

where the dimensional constant \mathcal{A} depends on the details, such as mass parameters and coupling constants, of the underlying field model, but does not depend on ϵ . The parameter μ in eq.(2) is the renormalized surface tension of the boundary separating the two vacuum phases in the limit $\epsilon \rightarrow 0$. It can be noted that the power of ϵ in the pre-exponent in eq.(2) is in agreement with the result of a direct calculation[13] in a specific ϕ^4 model. The claim in the present paper is that this power is universal in the limit $\epsilon \rightarrow 0$ so that it does not depend on details of specific model.

The calculation of the tunneling exponential factor as well as of the pre-exponent is conveniently done in terms of the Euclidean-space formulation of the model[7, 8]. The partition function Z for the metastable vacuum state is written in terms of the Euclidean action S in the standard form of the path integral:

$$Z = \mathcal{N} \int e^{-S[\phi]} \mathcal{D}\phi \quad (3)$$

where \mathcal{N} is the normalization factor and the path integration runs over all the fields in the model, generically denoted here as ϕ , with the boundary condition that the fields approach

¹It can be also mentioned that the resulting equation (1) in (1+1) dimensions has no corrections in powers of the dimensionless parameter ϵ/μ^2 . Only terms with higher exponents of $-\pi\mu^2/\epsilon$ are possible[9].

their values in the metastable vacuum at the boundaries of the normalization space-time box. The metastability of the considered ‘vacuum’ state results in that the partition function develops imaginary part (similar to the imaginary part of the energy of a resonance) which determines the decay rate (per unit space-time volume) according to the relation[10, 7] $\Gamma = 2\text{Im}(\ln Z)/(VT)$ with VT being the space-time volume of the normalization box.

The action functional $S[\phi]$ has a saddle point configuration corresponding to the tunneling trajectory of the bubble, the so-called bounce[7]. In the thin wall approximation the bounce is a three-dimensional ball of the ‘true’ vacuum separated from the bulk of the false vacuum by the thin wall with the surface tension μ . The radius R_c of the bounce is thus determined by the extremum of the effective action

$$S_{eff} = \mu A_B - \epsilon V_B , \quad (4)$$

where V_B is the volume of the ball and A_B is its surface area. Clearly, the extremum of the action is achieved at $R_c = 2\mu/\epsilon$, which is the critical bubble size in (2+1) dimensions, and the value of the action on the bounce is $S_B = B = (16\pi/3) \mu^3/\epsilon^2$, which gives the exponential factor in eq.(2).

In order to calculate the pre-exponential factor one has to evaluate the path integral Z_1 around the bounce configuration. The one-loop expression for the rate then reads as[8]

$$\Gamma = \frac{1}{VT} \left| \frac{\det(S_1^{(2)})}{\det(S_0^{(2)})} \right|^{-1/2} e^{-B} , \quad (5)$$

where $S_1^{(2)}$ is the operator of the second variation of the (Euclidean) action at the bounce configuration, and $S_0^{(2)}$ is the same operator for variation of the fields around the ‘flat’ false vacuum state. The operator $S_1^{(2)}$ has three translational zero modes and the integration over those cancels against the space-time normalization factor in the denominator. In Ref.[8] this integration has been done explicitly. However in the present discussion it is somewhat more convenient to leave the expression in the symbolic form (5) and to deal with the zero modes later. Also it should be mentioned that the operator $S_1^{(2)}$ has one negative mode, integration over which produces the imaginary part of the partition function, as explained in Ref.[8].

The spectrum of the operator $S_1^{(2)}$ consists of two substantially different parts: the hard part with the eigenvalues λ_n starting at the scale set by the mass parameters of the underlying field model, which scale is generically denoted here as m , and the soft part, whose scale is set by the (inverse) radius of the bounce. The soft part of the spectrum is universal and

does not depend on details of the underlying field model as long as the condition $mR_c \gg 1$ is satisfied, which is always the case in the thin wall limit, i.e. at $\epsilon \rightarrow 0$. The explicit expression for the soft eigenvalues is [8]

$$\lambda_\ell = \frac{\ell(\ell+1) - 2}{R_c^2}, \quad (6)$$

where $\ell = 0, 1, 2, \dots$ is the angular momentum and, obviously, the degeneracy of each mode is $2\ell + 1$. It can be readily noticed that the soft eigenmodes coincide (up to an overall normalization) with those found from the effective action (4). Indeed, using the parametrization of the surface of the bounce in polar coordinates as $r(\theta, \varphi)$ and expanding the radial position r around its value at the extremum of the action: $r(\theta, \varphi) = R_c + \sigma(\theta, \varphi)$, one gets in the quadratic order in the deviation σ the expression

$$S_{eff} = S_B + \frac{\mu}{2} \int \left(\partial_\alpha \sigma \partial^\alpha \sigma - 2\sigma^2 \right) d\Omega, \quad (7)$$

where $\partial_\alpha \sigma \partial^\alpha \sigma = (\partial_\theta \sigma)^2 + (\partial_\varphi \sigma)^2 / \sin^2 \theta$ and $d\Omega = \sin \theta d\theta d\varphi$. Clearly, the spectrum of the quadratic part in eq.(7) is proportional to that in eq.(6).

The described separation between the hard and the soft modes becomes ambiguous at $\ell \sim mR_c$, where the soft part merges into the hard one. The details of this merger would be unimportant if the path integral over the soft spectrum were convergent. Then the whole calculation would be reduced (by the Appelquist-Carazzone theorem [14]) to calculating the path integral with the effective action in eq.(4), i.e. in the effective ‘low-energy’ theory. This however is the case only in (1+1) dimensions [9], while in the discussed here (2+1) dimensional case the path integral with the effective action (4) diverges, and one has to resort to a somewhat more accurate application of the Appelquist-Carazzone theorem, which requires [14] an explicit consideration of the regularization of the effective low-energy theory.

In order to describe the regularization of the effective low-energy theory we concentrate now on the notion of the parameter μ in the limiting case of $\epsilon = 0$. At zero ϵ the two considered vacua are degenerate and there is a stable field configuration interpolating between them. In the three dimensional Euclidean space the interface between the vacua makes a two-dimensional surface with the action proportional to the area of the surface:

$$S_{eff} = \mu \cdot Area. \quad (8)$$

The presence of the wall spontaneously breaks the translational invariance. As a result there appears a spectrum of (Goldstone) modes, that can be described by a massless two-dimensional scalar propagating on the surface of the wall. The spectrum can also be readily

found from the effective action (8) by considering small deviations of the position of the wall from its equilibrium (flat) shape. On a flat wall the spectrum of these modes can be parametrized by the two-dimensional momentum k_α ($\alpha = 1, 2$): $\lambda_k = k^2$. A calculation of the partition function in the effective ‘low-energy’ theory at the one-loop level immediately runs into the problem that the integral over the Goldstone modes is divergent. In particular, the renormalization of μ in the effective theory is quadratically divergent:

$$\mu \rightarrow \mu + \frac{1}{2} \int \ln k^2 \frac{d^2 k}{(2\pi)^2} . \quad (9)$$

In the full theory (as opposed to the effective one) however, no such divergence arises, since the effective description fails at $k \sim m$, where the soft spectrum merges into the hard one. Thus physically the ultraviolet cutoff in the integral in eq.(9) is at the scale m , and the surface tension gets a one-loop quantum correction of order m^2 , which is the normal behavior in the full theory. In order to still enable a description of the low-energy modes by the effective action (8) at the loop level, it is necessary to explicitly introduce a separation parameter, which would serve as an ultraviolet regulator for the low-energy theory, while still being within the applicability of the expression for the soft modes. This can be done by using the standard Pauli-Villars regulator fields.

Following the Pauli-Villars procedure we introduce a set of regulator fields ψ_i (at least two are required to regularize the quadratic divergence in eq.(9)) with the mass parameters M_i . For each regulator field the spectrum of the eigenvalues is shifted up with respect to those of the fields in the original field model (λ_n) by M_i^2 :

$$\lambda_n(\psi_i) = \lambda_n + M_i^2 . \quad (10)$$

The loop with the regulator field ψ_i is given the weight factor c_i subject to the condition:

$$\sum_i c_i = 1 , \quad \sum_i c_i M_i^2 = 0 . \quad (11)$$

The regulator mass parameters M_i are assumed to be much less than the full theory mass scale: $M_i \ll m$, but much larger than the inverse size of the surface of the wall: $M_i \gg (Area)^{-1/2}$.

Formally, the described regulator fields are introduced in the path integral Z of the original theory by inserting a factor of one in the form:

$$Z = \frac{Z_\psi}{Z_\psi} Z , \quad (12)$$

where $Z_\psi = \prod_i (Z[\psi_i])^{c_i}$. The partition functions can then be split (at least at the one-loop level) into the products of the soft (Z_s) and hard (Z_h) contributions, e.g. $Z = Z_s Z_h$, where the separation between the “soft” and “hard” modes is introduced at a scale Λ , intermediate between M_i and m : $M_i \ll \Lambda \ll m$. Then the identity (12) for the partition function can be rewritten as

$$Z = \frac{Z_s}{(Z_\psi)_s} \{ (Z_\psi)_s Z_h \} . \quad (13)$$

Clearly, the first factor is the regularized partition function described by the effective action in eq.(8). In particular the regularized (at one loop) surface tension in this effective theory reads as

$$\mu_{reg} = \mu + \frac{1}{2} \int \left[\ln k^2 - \sum_i c_i \ln(k^2 + M_i^2) \right] \frac{d^2 k}{(2\pi)^2} = \mu + \frac{\bar{M}^2}{8\pi} , \quad (14)$$

where $\bar{M}^2 = \sum_i c_i M_i^2 \ln M_i^2$.

The expression in the curly braces in eq.(13) is the original path integral with the soft modes replaced by those of the regulator fields, which implies that all the modes relevant for calculation of the latter expression are hard in the sense that they start at least from the scale of regulator masses M_i . Thus at any shape of the wall the latter expression is sensitive only to local properties of the surface, i.e. to higher curvatures. In particular, if the wall is curved with a large radius R this part can produce corrections to the effective action in eq.(8) of at most the relative magnitude $O(M^{-2}R^{-2})$. In particular at the curvature corresponding to the radius R_c of the bounce such “finite wall thickness” corrections² are not singular in ϵ at $\epsilon \rightarrow 0$. For this reason the discussed consideration of the regularization procedure for a flat wall (at $\epsilon = 0$) is also applicable at the intended level of accuracy for a spherical wall of the bounce at a small but finite ϵ . In the latter case the effective low-energy action (8) can be replaced by the one in eq.(4) since the term with ϵ requires no regularization in the effective low-energy theory.

Thus the discussed problem of calculating the rate of the false vacuum decay is reduced to evaluating the contribution of the bounce configuration and of the fluctuations around it to the partition function determined by the effective action (4) and regularized by the ‘soft’ regulator factor $(Z_\psi)_s$. As already mentioned the value of the effective action (4) on the saddle point configuration reproduces the exponential factor in the decay rate, while the

²An expression for these corrections in a ϕ^4 model can be found in Ref.[13].

pre-exponential factor can be written as

$$F = f_0 \exp \frac{1}{2} \sum_{\ell=0}^{\infty} (2\ell+1) \left\{ \sum_i c_i \ln [\ell(\ell+1) + M_i^2 R_c^2 - \omega^2] - \ln [\ell(\ell+1) - \omega^2] \right\}, \quad (15)$$

where the constant f_0 comes from the expression in curly braces in eq.(13) and does not depend on ϵ (or, equivalently, on R_c), and the parameter ω is temporarily introduced in order to regularize the infrared singularity at $\omega^2 = 2$ due to the translational zero modes. One can also notice that, formally, the upper limit in the sums over ℓ should be set at a large value L related to the previously introduced separation parameter Λ as $L \sim \Lambda R_c$. However the overall sum in eq.(15) is convergent (at $\ell_{max} \sim M_i$) due to the Pauli-Villars constraints (11), and the sum can be extended to infinity as shown. Furthermore, the overall normalization factor in the eigenvalues is not important, since the total number of modes is the same for the regulator fields and the original soft part of the spectrum, so that any common additive term cancels in the total sum.

The sums in eq.(15) can be readily evaluated (up to a numerical additive constant) using the Euler-Maclaurin summation formula with the result reading (at ω^2 close to 2) as

$$F = \tilde{f}_0 \frac{(MR_c)^{\omega^2+1/3}}{|2 - \omega^2|^{3/2}} \exp \left(-\frac{1}{2} \bar{M}^2 R_c^2 \right), \quad (16)$$

where $\ln M = \sum_i c_i \ln M_i$, and the constant \tilde{f}_0 differs from f_0 in eq.(15) only by a numerical factor. One readily recognizes the term proportional to \bar{M}^2 in the exponent as the renormalization, according to eq.(14) of the surface tension μ in the effective action (4), and this term, together with the contribution of the hard modes (the expression in the curly brackets in eq.(13)) replaces the lowest-order surface tension μ by the one with the one-loop correction in the leading semiclassical exponent.

It is the factor with a power of MR_c in eq.(16) which produces a nontrivial power dependence of the pre-exponential factor F on the parameter ϵ , through the relation $R_c \propto \epsilon^{-1}$. Although one can safely set $\omega^2 = 2$ in this term, the dependence on ω is shown in eq.(16) in order to illustrate the origin of the contributions to this term: the ω^2 part comes from the shift of the eigenvalues with respect to $\ell(\ell+1)$, while the extra $1/3$ in the power is due to the discretization of the modes on a sphere. As will be discussed few lines below, a proper treatment of the denominator in eq.(16) singular at $\omega^2 = 2$ produces no extra dependence of the factor F on ϵ . Thus after setting $\omega^2 = 2$ in the power of MR_c one arrives at the formula (2) for the $\epsilon^{-7/3}$ behavior of the pre-exponential factor in the decay rate.

The singular at $\omega^2 = 2$ behavior arises in eq.(16) due to the space-time translational invariance in the probability of bubble nucleation. This singularity can be readily dealt with by either the standard consideration[8] of the integration over the translational zero modes, or by using the following simple regularization procedure in terms of the effective theory[9]. Let us temporarily slightly break the translational invariance by introducing a dependence of the nucleation probability on the position \vec{x} of the center of the bounce described by the Gaussian factor $\exp(-\xi x^2)$ with a small parameter ξ . The total probability of the nucleation in a large space-time volume is then finite and is given by

$$\int \Gamma e^{-\xi x^2} d^3x = \left(\frac{\pi}{\xi}\right)^{3/2} \Gamma. \quad (17)$$

On the other hand the shift of the center of the bounce is equivalent to an amplitude of the partial wave with $\ell = 1$ of the excitation σ of the surface of the bounce in eq.(7). Thus the effect of the introduced infrared regularization is equivalent to adding to the action (7) additional term $(3\xi/4\pi) \int \sigma^2 d\Omega$. Although, formally, this term should be added only for the partial wave with $\ell = 1$ where it lifts the translational modes from zero, it can be safely added to all modes, since for all other modes the limit $\xi \rightarrow 0$ is nonsingular. As a result the described infrared regularization is equivalent to the shift of the ‘frequency’ from $\omega^2 = 2$ to $\omega^2 = 2 - 3\xi/(2\pi\mu)$. Using this regularized expression in eq.(16) one readily finds from comparison with eq.(17) that the rate Γ (per unit space-time volume) is finite in the limit $\xi \rightarrow 0$, and no new dependence on ϵ of the factor F is introduced by the infrared regularization (only an extra dependence on μ does arise).

One can see from the presented here calculation that the pre-exponential factor $\epsilon^{-7/3}$ in the rate of false vacuum decay in (2+1) dimensions arises in fact as an analog of the Casimir effect on the finite surface of the bounce due to the massless spectrum of excitations of the surface waves (Goldstone modes) described by the effective action (4). As previously mentioned, in a (1+1) dimensional case this soft spectrum completely dominates and fully determines the relevant path integral for calculating the rate. In the considered here (2+1) dimensional case the importance of the soft part of the spectrum is greatly weakened in comparison, but is still sufficient for deriving the dependence on the large-scale parameter $R_c \propto \epsilon^{-1}$ of the pre-exponential factor. It further looks highly unlikely that in a (3+1) dimensional case the soft part of the spectrum alone can be used to make any conclusions about the behavior of pre-exponential factor in the rate of the false vacuum decay.

The presented here consideration, and the resulting formula in eq.(2) can be used essen-

tially without modification for the case of thermal decay of a metastable state in a first order phase transition in a three-dimensional system (i.e. in (3+1) dimensions) near the stability point, i.e. at $\epsilon \rightarrow 0$. However in realistic thermal systems the nucleation of the bubbles is usually governed by a diffusion dynamics, rather than by the conservative Hamiltonian one implied throughout the discussion in this paper. The pre-exponential factor in the nucleation rate in a diffusion dynamics case is a well known textbook material[15] going back to the original work of Zel'dovich[16].

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Note added. After this paper was completed I became aware of the work [17], where the behavior of the pre-exponential factor equivalent to $\epsilon^{-7/3}$ in (2+1) dimensions has been derived in the context of membrane creation by an antisymmetric tensor field. I believe that the calculation in the present paper is somewhat simpler, and is more directly related to the generic framework of one-loop calculations in specific field-theoretical models.

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